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On the Performance of the Linear Discriminant Function for Spherical Distributions

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A general integral expression is obtained for evaluating the performance of Fisher's linear discriminant function applied to spherical distributions (SD). Recurrence relations are given for certain special cases including the spherical gamma, Pearson VII, and generalized Laplace distributions. Some easily obtained upper and lower bounds for the probabilities of correct classification are shown to improve considerably the only available bounds, given by Haralick (*Pattern Recognition* **9**, 1977).

1. INTRODUCTION

In the simplest classification problem with two normal alternatives, the optimality properties of the well-known linear discriminant function (LDF) have been studied by several authors since its introduction by Fisher [13]. However, in general when the populations are non-normal, several difficulties arise, the main one being that the usual statistical (optimality) criteria do not lead to simple decision rules.

Usually, minimum-distance discrimination rules yield procedures which have certain optimality properties, no matter what kind of distance between the two populations is used. A general distance that can be employed for any two distributions is the Hellinger distance or, equivalently, Matusita's [20] affinity. If this happens to be a monotone function of the Euclidean or Mahalanobis distance between the population means, then it is feasible to use the LDF, obtained as the difference of the squared distances of $\mathbf{x}$, the observation to be classified, from the means $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ of the two alternative populations. This monotonicity property holds for

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several families of spherical distributions (SD), Cacoullos and Koutras [6]. Cooper [9, 10] proposed the same procedure for certain classes of SD's, having in mind as optimality criterion the minimization of the total probability of error. In any case, irrespective of optimality criteria, it may be of some interest in itself to examine the performance of the linear discriminant rule (LDR) for general SD's, of which the normal is a very special case. This is the purpose of the present paper.

2. Definitions and Preliminaries

A random column-vector $\mathbf{Y} = (Y_1, ..., Y_p)'$ is said to have a spherically symmetric or simply spherical distribution (SD) around the origin, if $\mathbf{Y}$ and $P\mathbf{Y}$ have the same distribution for every orthogonal $p \times p$ matrix P. If we consider the affine transformation $\mathbf{X} = \boldsymbol{\mu} + \Sigma^{1/2}\mathbf{Y}$, where $\boldsymbol{\mu}$ is a constant vector and Σ a positive definite matrix of order p, we obtain an elliptically symmetric distribution (ESD), with location parameter $\boldsymbol{\mu}$ and scale parameter Σ [17]. The density function (df) of $\mathbf{X}$, if it exists, has the form

$$f_{\mathbf{X}}(\mathbf{x}) = c_p \, |\Sigma|^{-1/2} \, g((\mathbf{x} - \boldsymbol{\mu})' \, \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})), \tag{2.1}$$

where $g(\cdot)$ is a non-negative real-valued function such that

$$\int_0^\infty t^{p-1} g(t^2) \, dt < \infty,$$

and c_p a normalizing constant. In such cases, we write $\mathbf{X} \sim \mathrm{ESD}(\boldsymbol{\mu}, \Sigma; g)$.

This paper is concerned with ESD's with densities and the same known Σ but different means $\boldsymbol{\mu}$. Hence we may assume that $\Sigma = I$ and restrict ourselves to the class of spherical distributions around $\boldsymbol{\mu}$, to be denoted by $\mathrm{SD}(\boldsymbol{\mu}; g)$.

3. Evaluation of the Probability of Correct Classification

Let Π_i: $\mathrm{SD}(\boldsymbol{\mu}_i; g)$, $i = 1, 2$, be two p-variate spherical distributions with known means $\boldsymbol{\mu}_i$, and common $g(\cdot)$ as in (2.1). The classification procedure considered in this paper is based on the difference of the squared Euclidean distances of the observation $\mathbf{X} = \mathbf{x}$ from $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2$. Thus letting $Q_i^2(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\mu}_i)' \, (\mathbf{x} - \boldsymbol{\mu}_i)$ $i = 1, 2$, $\mathbf{x}$ will be classified into Π_1 iff $\mathbf{x} \in R_1$, where

$$R_1 = \{\mathbf{x} \in R^p : Q_1^2(\mathbf{x}) \leqslant Q_2^2(\mathbf{x})\}. \tag{3.1}$$

This rule, which plays a primary role in standard normal discriminant analysis [4] has been proved to be also optimal for several large classes of SD's [9, 10, 6].

Relation (3.1) leads to the well-known linear discriminant function (LDF), whose distribution is difficult to obtain for non-normal SD's. Thus a method for evaluating the PCC becomes very useful.

Let $\mathbf{X} \sim \mathrm{SD}(\boldsymbol{\mu}_1, g)$. Then $Q_1^2(\mathbf{X}) = R^2$ follows a generalized chi-square distribution, $G\chi_p^2(g)$, with density [19],

$$f_{R^2}(t) = \frac{c_p \pi^{p/2}}{\Gamma(p/2)} t^{(p/2)-1} g(t), \qquad t > 0 \tag{3.2}$$

while $Q_2^2(\mathbf{X}) = R_\delta^2$ follows a non-central generalized chi-square distribution, $G\chi_p^2(\delta^2; g)$, with non-centrality parameter

$$\delta^2 = (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)' (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1). \tag{3.3}$$

For details see Anderson and Fang [3] and Cacoullos and Koutras [7]. It can be shown, Cacoullos and Koutras [7], that R_δ^2 is related to R^2 by

$$R_\delta^2 = R - 2R\delta \cos \Theta + \delta^2,$$

where Θ is a random angle, independent of R, with df

$$f_\Theta(\theta) = \sin^{p-2} \theta / B\left(\frac{1}{2}, \frac{p-1}{2}\right), \qquad 0 \leqslant \theta \leqslant \pi. \tag{3.4}$$

Hence

$$P(1/1) = \Pr[R \cos \Theta \leqslant \delta/2] = \tfrac{1}{2} + \Pr[R \cos \Theta \leqslant \delta/2,\ \Theta < \pi/2].$$

Finally, if we denote by $F_R(t)$ the cumulative distribution function (CDF) of the generalized chi, $G\chi_p(g)$, variable R, we easily obtain the probability of correct classification (PCC)

$$\begin{aligned} P = P(1/1) = P(2/2) &= \frac{1}{2} + \left(1 \Big/ B\left(\frac{1}{2}, \frac{p-1}{2}\right)\right) \int_0^{\pi/2} F_R(\delta/2 \cos \theta\Big) \sin^{p-2} \theta \, d\theta \\ &= \frac{1}{2} \left\{ 1 + F_R(\delta/2) + \int_{\delta/2}^{\infty} I_{\delta^2/4t^2}\left(\frac{1}{2}, \frac{p-1}{2}\right) f_R(t) \, dt \right\}, \end{aligned} \tag{3.5}$$

where $I_x(a, b)$ denotes the incomplete beta function ratio, Erdelyi *et al.* [12].

As expected, it follows from (3.5) that the PCC P is an increasing function of the Euclidean distance δ between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ and tends to 1 as $\delta \to \infty$. Moreover from (3.5) it is clear that

$$P \geqslant \tfrac{1}{2} + \tfrac{1}{2} F_R(\delta/2), \tag{3.6}$$

yielding Haralick's [14] lower bound for P.

4. Application to Specific Distributions

In this section we consider certain families of SD's and apply formula (3.5). Though direct calculations are difficult, it is very easy to obtain recurrence relations leading to a final computation of PCC. Some "optimality" properties of the normal distribution are also provided.

a. *Spherical Gamma*

SD($\boldsymbol{\mu}, g_n$) with

$$g_n(t) = t^{n-1} \exp(-rt), \qquad 2n+p-2>0, \qquad r>0. \tag{4.1}$$

It was introduced by Kotz [18] and includes, obviously, the spherical normal as a special case. If $F_n(t)$ denotes the CDF of $G\chi_p(g_n)$, it is easy to verify that

$$F_{n+1}(t) = F_n(t) - \frac{1}{\Gamma(n+p/2)} (rt^2)^{n+(p/2)-1} \exp(-rt^2),$$

and applying (3.5) we arrive at the recurrence

$$P_{n+1} = P_n - \frac{\Gamma(p/2)}{\sqrt{\pi}\,\Gamma(n+p/2-1)} (r\delta^2/4)^{n+(p/2)-1} \times \exp(-r\delta^2/4)\, \Psi\left(\frac{p-1}{2}, \frac{p-1}{2}+n; r\delta^2/4\right),$$

where Ψ denotes a confluent or degenerate hypergeometric function, Erdelyi *et al.* [12]. If n is a positive integer, we have

$$P_{n+1} = P_n - \frac{1}{2\sqrt{\pi}} \exp(-r\delta^2/4) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{[(p-1)/2]_k}{[p/2]_n} (r\delta^2/4)^{n-k-(1/2)}) \tag{4.2}$$

where

$$[a]_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1), \qquad a \in R,\ n = 0, 1, \ldots.$$

In the special case of a normal spherical distribution with $g(t) = \exp(-rt)$ we easily obtain that

$$P = \Phi(\delta(r/2)^{1/2}), \tag{4.3}$$

where $\Phi(\cdot)$ is the standard normal CDF.

Hence we conclude

(i) The PCC for the two-group LDR decreases monotonically with n. Its maximum value is given by (4.3) and is achieved by the spherical normal distribution ($n=1$).

(ii) For each fixed $n=1, 2, \ldots$, the PCC tends to the upper bound (4.3) as $p \to \infty$.

(iii) For fixed n and p, there exists $\delta_0 > 0$ such that the difference $P_n - P_{n+1}$ decreases monotonically with δ for $\delta > \delta_0$, and tends to 0 as δ approaches infinity.

b. *Spherical Pearson VII*

This term is used for the natural multivariate extension of the Pearsonian VII curve, with

$$g_n(t) = \left(1 + \frac{t}{s}\right)^{-n}, \qquad n > p/2. \tag{4.4}$$

It is clear that this family includes Dunnett and Sobel's [11] generalization of Student's t-distribution, as well as the multivariate Cauchy (see [15]).

If $F_n(t)$ denotes the CDF of $G\chi(g_n)$, a simple calculation by (3.2) and a well-known property of the incomplete beta function ratio (see, e.g., [5]) yield

$$F_n(t) = I_{t^2/(s+t^2)}\left(\frac{p}{2}, n - \frac{p}{2}\right),$$

$$F_{n+1}(t) = F_n(t) + \frac{\Gamma(n)\, s^{n-(p/2)}}{\Gamma(p/2)\, \Gamma(n-(p/2)+1)} \frac{t^p}{(s+t^2)^n}. \tag{4.5}$$

Finally, using (3.5), we arrive at

$$P_{n+1} = P_n + \frac{\Gamma(n-(p-1)/2)}{2\sqrt{\pi}\,\Gamma(n-(p/2)+1)} z^{1/2}(1-z)^{n-(p/2)}, \qquad z = \delta^2/(\delta^2+4s). \tag{4.6}$$

If we assume that $n = k/2$, $k > 2$ integer, it is evident that for the evaluation of P_n it suffices to calculate the starting values $P_{(p/2)+1}$, $P_{(p+1)/2}$ and then apply (4.6). Thus a straightforward calculation shows

$$P_{(p/2)+1} = \frac{1}{2}(1+\sqrt{z}), \qquad P_{(p+1)/2} = \frac{1}{2} + \frac{1}{\pi} \arcsin\sqrt{z}.$$

It is interesting to note, that, besides recurrence (4.6) an explicit expression for P_n may be derived. More specifically using (3.5) and expanding $F_n(t)$ of (4.5) in a power series we may prove that

$$P_n = \frac{1}{2} + \frac{z^{1/2}(1-z)^{n-(p/2)}}{B(1/2, n-(p/2))} F\left(1, n - \frac{p}{2} + \frac{1}{2}; \frac{3}{2}; z\right),$$

where F denotes the well-known Gauss hypergeometric function [12],

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{[c]_k} \cdot \frac{x^k}{k!}, \qquad |x| < 1, c \neq 0, -1, \ldots.$$

For the multivariate t-distribution with s degress of freedom $(n = (p+s)/2)$ the PCC becomes

$$P'_s = \frac{1}{2} + \frac{z^{1/2}(1-z)^{s/2}}{B(1/2, s/2)} F\left(1, \frac{s+1}{2}; \frac{3}{2}; z\right), \qquad z = z(s) = \delta^2/(\delta^2 + 4s)$$

Using the well-known expression of the CDF of the univariate t distribution with s degrees of freedom in terms of the Gauss function [2], we may prove that

$$P'_s(t) = \Pr[t_s \leqslant \delta/2] = F_{t_s}(\delta/2)$$

and the next assertions follow immediately.

1. For the multivariate t-distribution the PCC is an increasing function of the number s of degrees of freedom and obtains its maximum value in the limiting case $s \to \infty$, that is,

$$\sup\{P'_s : s = 1, 2, \ldots,\} = P'_\infty = \Phi(\delta/2).$$

2. The PCC for the multivariate t-distribution is independent of the dimensionality p.

c. *Generalized Laplace or Besel Distribution*

By this we mean the SD($\boldsymbol{\mu}$, g_n) with

$$g_n(t) = (\sqrt{t}/s)^n K_n(\sqrt{t}/s), \qquad n > -p/2, s > 0,$$

where $K_n(\cdot)$ denotes the modified Bessel function of the third kind or Macdonald's function [22]. It was derived first by Lord [19] by summing up random vectors following a p-dimensional exponential distribution and later by Miller [21] who considered certain products of Gaussian variates.

On using the well-known identity [12],

$$K_{n+1}(t) = \frac{n}{t} K_n(t) - \frac{d}{dt} K_n(t),$$

we may easily prove that the CDF $F_n(t)$ of $G\chi(g_n)$ satisfies the recurrence

$$F_{n+1}(t)=F_n(t)-\frac{1}{2^{n+p-1}\Gamma(p/2)\,\Gamma(n+(p/2)+1)}\left(\frac{t}{s}\right)^{n+p}K_n\left(\frac{t}{s}\right)$$

from which we obtain

$$P_{n+1}=P_n-\frac{1}{\sqrt{\pi}\Gamma(n+(p/2)+1)}\left(\frac{\delta}{4s}\right)^{n+(p+1)/2}K_{n+(p-1)/2}\left(\frac{\delta}{4s}\right). \quad (4.7)$$

In particular for $n=n_0=(-p/2)+1$ the CDF and corresponding F_n and P_n reduce to

$$F_{n_0}(t)=1-\frac{2}{2^{p/2}\Gamma(p/2)}\left(\frac{t}{s}\right)^{p/2}K_{p/2}\left(\frac{t}{s}\right), \qquad P_{n_0}=1-\frac{1}{2}\exp\left(-\frac{\delta}{2s}\right). \quad (4.8)$$

Notice that formulae (4.7) and (4.8) provide an easy way of computing P_n for $n=-p/2+k$, $k=1, 2, \ldots$.

5. Some Bounds of the PCC for the LDR

Although, in most cases, it is easy to derive an expression for the CDF of $G\chi(g)$, we can rarely express the integral (3.5) in closed form. Thus, it is useful to obtain some computationally convenient bounds, especially lower ones for the PCC and discuss their application to specific spherical families.

First note that, by (3.1), we have

$$R_1=\{\mathbf{x}\in R^p\colon (\mathbf{x}-\boldsymbol{\mu}_1)'\,\mathbf{c}\leqslant \delta^2/4\},$$

where $\mathbf{c}=(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)/2$ and therefore the PCC, P, is given by

$$P=\Pr[\mathbf{X}\in R_1/\Pi_1]=\Pr[\mathbf{Y}\in A/\mathbf{Y}\sim \mathrm{SD}(\mathbf{0};g)],\ A=\{\mathbf{y}\in R^p\colon \mathbf{y}'\mathbf{c}\leqslant \delta^2/4\}.$$

Now let $\{s_i\}$ be an increasing sequence of real numbers such that $s_0=0$, $s_1=\delta/2$ and define the following subsets of the p-dimensional Euclidean space

$$A_0=\{\mathbf{y}\in R^p\colon \mathbf{y}'\mathbf{c}\leqslant 0\},$$
$$A_i=\{\mathbf{y}\in R^p\colon s_{i-1}\leqslant |\mathbf{y}|<s_i,\ 0<(\mathbf{y}/|\mathbf{y}|)'\,\mathbf{c}\leqslant \delta^2/4s_i\},\ i\geqslant 1,$$

where $|\ |$ denotes the usual Euclidean norm in R^p. For any positive integer N, it is easy to verify that

$$A\supset\bigcup_{i=0}^{\infty}A_i\supset\bigcup_{i=0}^{N}A_i, \qquad A_i\cap A_j=\varnothing \quad \text{for} \quad i\neq j,$$

from which we obtain

$$P \geqslant \sum_{i=0}^{\infty} p_i \geqslant \sum_{i=0}^{N} p_i \equiv \mathrm{LB}_N, \qquad p_i = \Pr[\mathbf{Y} \in A_i],\ i=0, 1, 2, \ldots \tag{5.1}$$

Hence, by integrating the df of the SD over the $N+1$ non-overlapping areas A_i we can derive a lower bound LB_N for P. The advantage of the approximation involved in (5.1) is that the probabilities p_i can be easily evaluated, since

(a) A_0 is a half-space and A_1 is a half-sphere with radius $\delta/2$.

(b) A_i is the intersection of the exterior of the hyperspherical half-cone with vertex at the origin, axis passing through $\mathbf{c}$ and angle between the axis and a generator $\theta_i = \arccos(\delta/2s_i)$, with the spherical half-ring defined by two hyperspheres of radius s_{i-1} and s_i which are centered at the origin (see Fig. 1).

Hence, by the radial symmetry of A_i (see also [8, 16]),

$$p_0 = \tfrac{1}{2}, \qquad p_1 = \Pr[|\mathbf{Y}| < \delta/2, \mathbf{Y}'\mathbf{c} > 0] = \tfrac{1}{2}\Pr[|\mathbf{Y}| < \delta/2] = \tfrac{1}{2}F_R(\delta/2)$$

$$p_i = (F_R(s_i) - F_R(s_{i-1}))\, I_{\delta^2/4s_i^2}(1/2, (p-1)/2), \qquad i \geqslant 1.$$

Thus the following increasing sequence of lower bounds for the PCC results from (5.1)

$$\mathrm{LB}_0 = \frac{1}{2}$$

$$\mathrm{LB}_N = \frac{1}{2}\left\{1 + \sum_{i=1}^{N} I_{\delta^2/4s_i^2}(\tfrac{1}{2}, (p-1)/2)(F_R(s_i) - F_R(s_{i-1}))\right\}, \quad N \geqslant 1. \tag{5.2}$$

In the special case $N=1$, (5.2) yields Haralick's lower bound [14]. It is interesting to note that (5.2) can be written in the form

$$\mathrm{LB}_N = \frac{1}{2}\left\{1 + \sum_{i=1}^{N} F_R(s_i)(I_{\delta^2/4s_i^2}(\tfrac{1}{2}, (p-1)/2) - I_{\delta^2/4s_{i+1}^2}(\tfrac{1}{2}, (p-1)/2))\right\}, \tag{5.3}$$

where s_i are as previously, and by convention, $s_{N+1} = \infty$. A direct proof of the last formula is obtained by observing that the ith term of the sum is the probability content, under $\mathrm{SD}(\mathbf{0}; g)$, of the region which is enclosed between two hyperspherical half-cones with vertex at the origin, axis passing through $\mathbf{c}$ and angles $\arccos \delta/2s_i$, $\arccos \delta/2s_{i+1}$, and a hypersphere of radius s_i centered also at the origin.

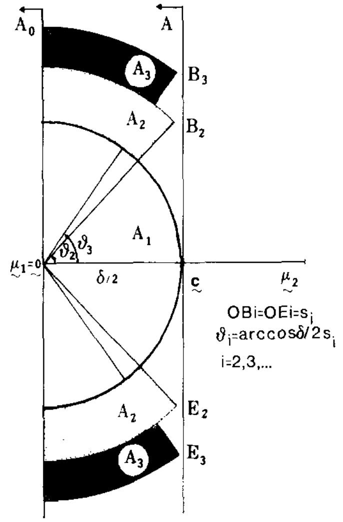

FIGURE 1

The preceding arguments can be properly modified to yield the upper bounds for P:

$$P \leqslant \frac{1}{2}\left\{1 + F_R(\delta/2) + \sum_{i=2}^{N} I_{\delta^2/4s_{i-1}^2}\left(\frac{1}{2}, \frac{p-1}{2}\right)(F_R(s_i) - F_R(s_{i-1}))\right\} \quad (5.4)$$

$$P \leqslant \frac{1}{2}\left\{1 + \sum_{i=2}^{N} F_R(s_i)\left(I_{\delta^2/4s_{i-1}^2}\left(\frac{1}{2}, \frac{p-1}{2}\right) - I_{\delta^2/4s_i^2}\left(\frac{1}{2}, \frac{p-1}{2}\right)\right)\right\}. \quad (5.5)$$

TABLE I

PCC for Π_i: SD$(\cdot, g)$, $g(t) = \exp(-t/2)$, $i = 1, 2$

p	δ	Δs	P (exact value)	Haralick's bound	LB_i $i=5$	LB_i $i=10$
2	1	0.25	0.6915	0.5588	0.6470	0.6710
	2	0.25	0.8413	0.6967	0.8060	0.8232
	4	0.50	0.9772	0.9323	0.96897	0.9690
3	1	0.25	0.6915	0.5154	0.6137	0.6700
	2	0.25	0.8413	0.5994	0.7715	0.8206
	4	0.50	0.9772	0.8693	0.9653	0.9655
10	1	0.50	0.6915	0.5000	0.5488	0.6767
	2	0.50	0.8413	0.5000	0.6726	0.8216
	4	0.50	0.9772	0.5263	0.9280	0.9685

TABLE II

PCC for Π_i: SD$(\cdot, g)$, $g(t) = t \exp(-t/2)$, $i = 1, 2$

p	δ	Δs	P (exact value)	Haralick's bound	LB_i $i=5$	LB_i $i=10$
2	1	0.25	0.6035	0.5036	0.5451	0.5891
	2	0.25	0.7203	0.5451	0.6533	0.7013
	4	0.25	0.9232	0.7969	0.9060	0.9116
3	1	0.50	0.6328	0.5008	0.5943	0.6162
	2	0.50	0.7606	0.5187	0.7167	0.7316
	4	0.50	0.9412	0.7253	0.9182	0.9197
10	1	0.50	0.6740	0.5000	0.5225	0.6604
	2	0.50	0.8171	0.5000	0.6070	0.7990
	4	0.50	0.9664	0.5083	0.8788	0.9560

It should be mentioned that the LB_N of (5.3) corresponds to a lower Darboux sum of the Riemann–Stieltjes inegral of $h(\theta) = F_R(\delta/2 \cos \theta)$ with respect to $F_\Theta(\cdot)$ over $[0, \pi/2]$. Similarly, (5.2) and (5.4) are connected with a Riemann–Stieltjes integral with respect to $F_R(t)$ over $[\delta/2, \infty)$.

In Tables I, II, and III our lower bound (after 5 and 10 steps) and Haralick's bound are compared with the exact value P of PCC, in the special cases of a normal, a spherical gamma, and a Pearson VII distribution. In the computations the s_i of formula (5.2) were chosen to be

TABLE III

PCC for Π_i: SD$(\cdot, g)$, $g(t) = (1+t)^{-n}$, $n = p/2 + 1$, $i = 1, 2$

p	δ	Δs	P (exact value)	Haralick's bound	LB_i $i=5$	LB_i $i=10$
2	1	0.25	0.7236	0.6000	0.6838	0.6987
	2	0.50	0.8536	0.7500	0.8249	0.8302
	10	1.00	0.9903	0.9808	0.9877	0.9888
3	1	0.25	0.7236	0.5447	0.6625	0.6930
	2	0.50	0.8536	0.6768	0.8117	0.8238
	10	1.00	0.9903	0.9714	0.9856	0.9880
10	1	0.50	0.7236	0.5002	0.6361	0.6906
	2	0.50	0.8536	0.5156	0.7491	0.8172
	10	1.00	0.9903	0.9110	0.9690	0.9831

$s_i = \delta/2 + (i-1)\,\Delta s$, $i = 1, 2, \ldots$ Notice that, for large values of p, Haralick's bound is near 0.5, i.e., it is non-informative and misleading for the goodness of the LDR.

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